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Complete Sequences in $\mathbb{N}^2$

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Folkman's theorem states that if $A=\{a_1<a_2<\cdots\}$ satisfying $A(n)>n^{\frac{1}{2}+\varepsilon}$, where $A(n)=\sum_{a_i\leqslant n}1$, then $P(A)=\{a_{i_1}+a_{i_2}+\cdots+a_{i_s}: i_1<i_2<\cdots<i_s;\ s\in\mathbb{N};\ a_{ij}\in A\}$ contains an infinite arithmetic progression. We shall consider the analogue of Folkman's theorem in $\mathbb{N}^2$, and some related questions are also investigated.

1. Introduction

In 1966, J. Folkman proved that if $A=\{a_1<a_2<\cdots<a_n\cdots\}$ is a strictly increasing sequence of integers satisfying $A(n)>N^{\frac{1}{2}+\alpha}$ for some $\alpha>0$ (where $A(n)=\sum_{a_i\leqslant n}1$), then the set $P(A)=\{\sum \varepsilon_i a_i \mid \varepsilon_i=0 \text{ or } 1; \sum\varepsilon_i<\infty\}$ contains an infinite arithmetic progression. Let us call such an A subcomplete. Furthermore, if $P(A)$ intersects every arithmetic progression, then $P(A)$ contains all sufficiently large integers. Let us say then that A is complete.

In this paper we are going to investigate the analogous question in $\mathbb{N}^2$.

We shall prove that the analogue theorem of Folkman's theorem is not true, and we give some other results.

2. Notations and Definitions

Let us use the following notation. Let $\mathbf{v}\in\mathbb{N}^2$; $\mathbf{v}=(v_1,v_2)$, $v_1,v_2\in\mathbb{N}$. We shall sometimes write $v^{(1)}$ $(v^{(2)})$ for v_1 (v_2). Let

$$m(\mathbf{v})=\begin{cases} v_2/v_1 & \text{if } v_1\neq 0,\\ \infty & \text{if } v_1=0.\end{cases}$$

Let $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{N}^2$ and let

$$\mathbf{v}_1+\mathbf{v}_2=(v_1^{(1)}+v_2^{(1)},\ v_1^{(2)}+v_2^{(2)})$$

For $\mathbf{A},\subseteq\mathbb{N}^2$, denote by $\mathbf{A}+\mathbf{B}=\{\mathbf{x}\mid \mathbf{x}=\mathbf{a}+\mathbf{b}$ where $\mathbf{a}\in\mathbf{A}$ and $\mathbf{b}\in\mathbf{B}\}$. Let $\mathbf{v}\in\mathbb{N}^2$. Denote by $|\mathbf{v}|$ the length of $\mathbf{v}$, i.e. $|\mathbf{v}|=\sqrt{v_1^2+v_2^2}$. Let

$$\mathbf{S}_n=\{\mathbf{v}\mid \mathbf{v}\in\mathbb{N}^2 \text{ and } |\mathbf{v}|\leqslant n\}.$$

Let $\mathbf{A}=\{\mathbf{a}_1,\mathbf{a}_2,\ldots\}\subseteq\mathbb{N}^2$ and let $\mathbf{A}^{(n)}=\{\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_n\}$. We define the counting function of $\mathbf{A}$ as $\mathbf{A}(n)=|\mathbf{A}\cap\mathbf{S}_n|$, where $|\mathbf{X}|$ is the cardinality of a finite set $\mathbf{X}$.

An analogous notion of an interval is the discrete rectangle. Let

$$\mathbf{R}(a,b,c,d)=\{\mathbf{v}\mid \mathbf{v}\in\mathbb{N}^2, a\leqslant v_1\leqslant c \text{ and } b\leqslant v_2\leqslant d\}$$

and let $t(\mathbf{R})$ be the size of $\mathbf{R}(a,b,c,d)$, i.e. $t(\mathbf{R})=(c-a+1)(d-b+1)$. For given $\mathbf{x},\mathbf{d}\in\mathbb{N}^2$, call the set

$$L(\mathbf{x}_0,\mathbf{d})=\{\mathbf{x}_0+k\mathbf{d}\mid k\in\mathbb{N}\}$$

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an arithmetic progression. Let $L(\mathbf{d}) := L(\mathbf{0}, \mathbf{d})$. For a given set $\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \ldots\} \subseteq \mathbb{N}^2$, let

$$P(\mathbf{A}) = \{\textstyle\sum \varepsilon_i \mathbf{a}_i \mid \varepsilon_i = 0 \text{ or } 1; \sum \varepsilon_i < \infty\}.$$

Let us call $\mathbf{A}$ L-complete if there exist $\mathbf{x}_0, \mathbf{d} \in \mathbb{N}^2$ such that $L(\mathbf{x}_0, \mathbf{d}) \subset P(\mathbf{A})$.

We can define another completeness, as well. Let us call $\mathbf{A} \subseteq \mathbb{N}^2$ p-complete if there exist $\mathbf{x}_0 \in \mathbb{N}^2$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, $m(\mathbf{u}) \neq m(\mathbf{v})$, such that

$$S(\mathbf{x}_0, \mathbf{u}, \mathbf{v}) := \{\{\mathbf{x}_0 + \alpha\mathbf{u} + \beta\mathbf{v} \mid \alpha, \beta \in \mathbb{R}^+\} \cap \mathbb{N}^2\} \subset P(\mathbf{A}).$$

3. The Plane Analogue of Folkman's Theorem

In [2], the following remarkable result is established:

Theorem F (Folkman). *If A is an infinite set of positive integers such that there exist $\varepsilon > 0$ and n_0 with*

$$A(n) > n^{\frac{1}{2}+\varepsilon}$$

for $n > n_0$, then A is subcomplete.

In this section we prove that the analogous theorem does not hold for a sequence $\mathbf{A} \subseteq \mathbb{N}^2$.

Theorem 3.1. *There exist an $\alpha > 0$ and an $\mathbf{A} \subseteq \mathbb{N}^2$ for which*

$$\mathbf{A}(n) > \alpha n^2$$

and $\mathbf{A}$ is not L-complete.

Proof of Theorem 3.1. Let $0 < \gamma < \pi/4$ and $\varepsilon = \tan\gamma$. Let

$$\mathbf{R}_{2n-1} := \{\mathbf{x} \mid \mathbf{x} \in \mathbb{N}^2; 0 < m(\mathbf{x}) < \varepsilon \text{ and } 2^{4^{2n-1}} \leqslant |\mathbf{x}| < 2^{4^{2n}}\}$$

and

$$\mathbf{R}_{2n} := \{\mathbf{x} \mid \mathbf{x} \in \mathbb{N}^2; 1/\varepsilon < m(\mathbf{x}) < \infty \text{ and } 2^{4^{2n}} \leqslant |\mathbf{x}| < 2^{4^{2n+1}}\}$$

$n = 1, 2, \ldots$.

Now let $\mathbf{A} := \bigcup_{n=1}^{\infty} \mathbf{R}_n$.

First we prove that $\mathbf{A}(n) > \alpha n^2$ for some $\alpha > 0$, $n \in \mathbb{N}$. To prove it, let

$$(y_1, y_2) \in \mathbf{R}'_{2n} \text{ iff } (y_2, y_1) \in \mathbf{R}_{2n}, \qquad n \in \mathbb{N}.$$

Clearly, $|\mathbf{R}_{2n}| = |\mathbf{R}'_{2n}|$ for every $n \in \mathbb{N}$. Furthermore,

$$\bigcup_{n=1}^{\infty} (\mathbf{R}_{2n-1} \cup \mathbf{R}'_{2n}) = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{N}^2; 0 < m(\mathbf{x}) < \varepsilon\},$$

so we have

$$\mathbf{A}(N) = |\mathbf{A} \cap \mathbf{S}_N| = \left|\bigcup_{n=1}^{\infty} (\mathbf{R}_{2n-1} \cup \mathbf{R}'_{2n}) \cap \mathbf{S}_N\right| \geqslant (\gamma/2)\, |\mathbf{S}_N|/(\pi/2) = \alpha N^2.$$

Now we show that $P(\mathbf{A})$ does not contain an arithmetic progression. Let us assume that, contrary to the assertion, there are $\mathbf{x}_0, \mathbf{d} \in \mathbb{N}^2$ such that $L(\mathbf{x}_0, \mathbf{d}) \subset P(\mathbf{A})$. Let $|\mathbf{d}| = d$.

We distinguish two cases: $0 \leqslant m(\mathbf{d}) < 1$ or $1 \leqslant m(\mathbf{d}) \leqslant \infty$.

The proofs for the cases $0 \leq m(\mathbf{d}) \leq 1$ and $1 \leq m(\mathbf{d}) \leq \infty$ are similar. Thus, without loss of generality, let $0 \leq m(\mathbf{d}) \leq 1$ and $0 \leq m(\mathbf{x}_0) \leq 1$.

LEMMA 3.1. *Let*

$$\mathbf{E}_n := \{\mathbf{x} \mid \mathbf{x} \in \mathbb{N}^2; 4 \cdot \sqrt{2} \cdot 2^{3 \cdot 4^{2n}}/(1-\varepsilon) < |\mathbf{x}| < 2^{4^{2n+1}} \text{ and } 0 \leq m(x) \leq 1\}, \qquad n \in \mathbb{N}.$$

Then $\mathbf{E}_n \not\subset P(A)$.

PROOF OF LEMMA 3.1. Let

$$\mathbf{F}_n := \{\mathbf{x} \mid \mathbf{x} \in \mathbb{N}^2; 2^{4^{2n}} < |\mathbf{x}| < 2^{4^{2n+1}} \text{ and } 0 \leq m(x) < 1\}$$

and let $\mathbf{x} \in \mathbf{F}_n \cap P(\mathbf{A})$.

If $\mathbf{a}_i \in \bigcup_{k=n}^{\infty} \mathbf{R}_{2k+1}$, then $|\mathbf{a}_i| > 2^{4^{2n+1}}$. Thus, writing

$$\mathbf{x} = \mathbf{a}_{i_1} + \mathbf{a}_{i_2} + \cdots + \mathbf{a}_{i_s},$$

we have $\mathbf{a}_{i_j} \in \mathbf{U} \cup \mathbf{V}$, where $\mathbf{U} := \bigcup_{k=1}^{n} \mathbf{R}_{2k-1}$ and $\mathbf{V} := \bigcup_{k=1}^{\infty} \mathbf{R}_{2k}$, which implies

$$\mathbf{F}_n \cap P(\mathbf{A}) \subset P(\mathbf{U} \cup \mathbf{V}) = P(\mathbf{U}) + P(\mathbf{V}).$$

If $\mathbf{z} \in P(\mathbf{U})$; $\mathbf{z} = \mathbf{a}_{j_1} + \mathbf{a}_{j_2} + \cdots + \mathbf{a}_{j_t}$ then, by the triangle inequality, we obtain

$$|\mathbf{z}| \leq \sum_{r=1}^{t} |\mathbf{a}_{j_r}| \leq \left| \bigcup_{k=1}^{n} R_{2k-1} \right| \cdot \max |\mathbf{a}_{j_r}| < 4 \cdot 2^{2 \cdot 4^{2n}} \cdot 2^{4^{2n}} = 4 \cdot 2^{3 \cdot 4^{2n}}. \tag{3.1.1}$$

Now let

$$\mathbf{K} := \{\mathbf{x} \mid \mathbf{x} \in \mathbb{N}^2; 1/\varepsilon \leq m(\mathbf{x}) < \infty\}.$$

Clearly, $P(\mathbf{V}) \subset \mathbf{K}$, so by (3.1.1) we have

$$\mathbf{F}_n \cap P(\mathbf{A}) \subset \{\mathbf{x} \mid 0 \leq m(\mathbf{x}) < \varepsilon; |\mathbf{x}| < 4 \cdot 2^{3 \cdot 4^{2n}}\} + \mathbf{K}. \tag{3.1.2}$$

This implies by an easy calculation that $2^{4^{2n+1}} > |\mathbf{x}| > 4 \cdot \sqrt{2} \cdot 2^{3 \cdot 4^{2n}}/(1-\varepsilon)$ and $0 \leq m(\mathbf{x}) \leq 1$. Then $\mathbf{x} \notin \mathbf{F}_n \cap P(\mathbf{A})$, which proves the lemma. ■

Since

$$\Delta_n := 2^{4^{2n+1}} - 4 \cdot \sqrt{2} \cdot 2^{3 \cdot 4^{2n}}/(1-\varepsilon) \to \infty$$

if $n \to \infty$, there is an n_0 such that, for $n > n_0$,

$$\Delta_n > d.$$

Let us write $\mathbf{x}_k = \mathbf{x}_0 + k \cdot \mathbf{d} = (x_{k1}, x_{k2})$. Then, for every k, $0 \leq m(\mathbf{x}_k) \leq 1$ and $x_{k+1,1} - x_{k,1} < d$, respectively. We obtain that, for every $n > n_0$, there is a k such that $\mathbf{x}_k \in \mathbf{E}'_n \not\subset P(\mathbf{A})$. This contradiction proves the theorem. □

4. SEQUENCES WITH BOUNDED GAPS

If A is a sequence of positive integers, then there is an obvious sufficient condition for its completeness: if $A = A_1 \cup A_2$, $A_1 \cap A_2 = \varnothing$, $P(A_1)$ has bounded gaps (i.e. there is a $K > 0$ such that if $P(A_1) = \{p_1 < p_2 < \cdots\}$, then, for every n, $p_{n+1} - p_n < k$), and $P(A_2)$ contains sufficiently large intervals then A is complete. What happens if $A \subset \mathbb{N}$ is replaced by $\mathbf{A} \subset \mathbb{N}^2$?

DEFINITION 4.1. We say that $\mathbf{X} \subseteq \mathbb{N}^2$ has bounded gaps if there is a sequence $\{\mathbf{x}_i\}$ of distinct elements of $\mathbf{X}$ such that the differences $\mathbf{x}_{i+1} - \mathbf{x}_i$ assume only finitely many different values.

We prove the following theorem.

THEOREM 4.1. *There is an* $\mathbf{A} \subseteq \mathbb{N}^2$ *for which* $\mathbf{A} = \mathbf{A}_1 \cup \mathbf{A}_2$; $\mathbf{A}_1 \cap \mathbf{A}_2 = \emptyset$ *and*:
(1) *for every* $r, s \in \mathbb{N}$, *there are* $a, b \in \mathbb{N}$ *such that*

$$\mathbf{R}(a, b, a+r, b+s) \subset P(\mathbf{A}_1); \tag{4.1}$$

(2) $P(\mathbf{A}_2)$ *has bounded gaps*;
and $\mathbf{A}$ *is not L-complete.*

This $\mathbf{A}$ will have the following property:

$$\mathbf{a}_{n+1} - \mathbf{a}_n \in \{(1, 0), (0, 1)\} \qquad \text{for every } n \in \mathbb{N}. \tag{4.2}$$

We say that a sequence $\mathbf{A}$ is simple if (4.2) holds.
We mention that simple sequences have many interesting combinatorial properties (see [1] and [3]). Although a simple sequence $\mathbf{A}$ is not always complete, the set $P(\mathbf{A})$ contains a large regular part.

DEFINITION 4.2. Let $\mathbf{A} \subseteq \mathbb{N}^2$ be a simple sequence and let

$$T_A(n) := \max\{t(\mathbf{R}) \mid \mathbf{R} \text{ is a discrete rectangle and } \mathbf{R} \subset P(\mathbf{A}^{(n)})\}.$$

If $\mathbf{a}_1 = (1, 2)$ and $\mathbf{a}_{n+1} - \mathbf{a}_n = (1, 0)$ for $n \geqslant 1$, then it is not hard to see that $T_A(n) = n$.
Let $\mathbf{A} \subseteq \mathbb{N}^2$ and let us assume that

$$|\{i : \mathbf{a}_{i+1} - \mathbf{a}_i = (1, 0)\}| = |\{j : \mathbf{a}_{j+1} - \boldsymbol{a}_j = (0, 1)\}| = \infty. \tag{4.3}$$

We shall prove that if (4.3) holds, then $T(n)$ is much larger than n.

THEOREM 4.2. (1) *Let* $\mathbf{A} \subseteq \mathbb{N}^2$ *be a simple sequence. If* $\mathbf{A}$ *satisfies* (4.3) *and n is large enough, then*

$$n^{\frac{3}{2}}/360 < T_A(n).$$

(2) *On the other hand, there is a simple sequence* $\mathbf{A}$ *which satisfies* (4.3) *and*

$$T_A(n) < 4n^2.$$

Finally in this section, we give a sufficient condition for the p-completeness of $\mathbf{A} \subseteq \mathbb{N}^2$.

THEOREM 4.3. *Let* $\mathbf{X} \subseteq \mathbb{N}^2$. *Suppose that there are* $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{N}^2$, $m(\mathbf{d}_1) \neq m(\mathbf{d}_2)$ *and* $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$, *for which*

$$|L(\mathbf{d}_i) \cap \mathbf{X} \cap \mathbf{S}_n| > n^{\frac{1}{2}+\varepsilon}, \tag{4.4}$$

for $i = 1, 2$.

Assume that $\mathbf{A} := \mathbf{X} \backslash (L(\mathbf{d}_1) \cup L(\mathbf{d}_2))$ *satisfies* (4.3). *Then* $\mathbf{X}$ *is* p-*complete.*

First, we need some preliminary lemmas.

DEFINITION 4.3. We write

$$\mathscr{H}(B, S, A) := \{(u, v) \mid v = A;\ B < u \leqslant B + S\},$$
$$\mathscr{V}(A, S, B) := \{(u, v) \mid u = A;\ B < v \leqslant B + S\},$$

where $A, B, S \in \mathbb{N}$, and let

$$P_2(\mathbf{B}) := \{x_2 \mid \mathbf{x} \in P(\mathbf{B}),\ \mathbf{x} = (x_1, x_2),\ \mathbf{B} \subseteq \mathbb{N}^2\}.$$

Let

$$H(n) := |\{i : 2 \leqslant i \leqslant n;\ \mathbf{a}_i = \mathbf{a}_{i-1} + (1, 0)\}|\,,$$
$$V(n) := |\{j : 2 \leqslant j \leqslant n;\ \mathbf{a}_j = \mathbf{a}_{j-1} + (0, 1)\}|\,.$$

LEMMA 4.1. *Let* $\mathbf{A}$ *be an arbitrary sequence of* $\mathbb{N}^2$. *Let* $\mathbf{A}' = \{\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, \ldots, \mathbf{a}_{i_t}\}$ *be a* (*finite*) *subsequence of* $\mathbf{A}$. *Suppose that for each* j, $1 \leqslant j \leqslant t$, *we have* $\mathbf{a}_{i_j} - \mathbf{a}_{i_{j-1}} = (1, 0)$ (*or* $(0, 1)$). *Then there are* $A, B \in \mathbb{N}$ *such that* $P(\mathbf{A}) \supset \mathscr{H}(A, t/2, B)$ (*or* $P(\mathbf{A}) \supset \mathscr{V}(A, t/2, B)$).

PROOF OF LEMMA 4.1. Let $\mathbf{A}'' := \{\mathbf{a}_{i_{2j}} \mid \mathbf{a}_{i_{2j}} \in \mathbf{A}',\ j = 1, 2, \ldots [t/2]\}$. Let $\mathbf{L}_1 := \{\mathbf{a}_{i_1}\}$; $A_1 = a^{(2)}_{i_1}$, $B_1 = a^{(1)}_{i_1}$ and $D_1 = 1$ and, for $1 \leqslant j \leqslant [t/2]$, let

$$\mathbf{L}_j := \{L_{j-1} + \mathbf{a}_{i_{2j}}\} \cup \{\mathbf{L}_{j-1} + \mathbf{a}_{i_{2j-1}}\}$$

and $A_j = A_{j-1} + a^{(2)}_{i_{2j-1}}$; $B_j = B_{j-1} + a^{(1)}_{i_{2j-1}}$ and $D_j = D_{j-1} + 1$.

We claim that, for each j, $1 \leqslant j \leqslant [t/2]$,

$$\mathscr{H}(B_j, D_j, A_j) \subset \mathbf{L}_j. \tag{4.5}$$

For $j = 1$, (4.5) is just the definition. Let $j > 1$. If $(u, v) \in \mathbf{L}_{j-1}$ then, by

$$(\mathbf{a}_{i_{2j}} + (u, v)) - (\mathbf{a}_{i_{2j-1}} + (u, v)) = (1, 0),$$

we obtain

$$\mathbf{L}_j \supset \mathscr{H}(B_{j-1} + a^{(1)}_{i_{2j-1}}, D_{j-1} + 1, A_{j-1} + a^{(2)}_{i_{2j-1}}) = \mathscr{H}(B_j, D_j, A_j).$$

Let $j = [t/2]$. Then $D_{[t/2]} = [t/2]$ and thus

$$P(\mathbf{A}) \supset \mathbf{L}_{[t/2]} \supset \mathscr{H}(B_{[t/2]}, [t/2], A_{[t/2]}). \qquad \square$$

LEMMA 4.2. *Let* $H_1, H_2 \subset \mathbb{N}^2$; $\mathbf{H}_1 \cap \mathbf{H}_2 = \varnothing$. *Suppose that*

$$\mathscr{H}(A_1, D_1, B_1) \subset P(\mathbf{H}_1) \text{ and } \mathscr{V}(A_2, D_2, B_2) \subset P(\mathbf{H}_2).$$

Then there are $A, B \in \mathbb{N}$, *such that*

$$\mathbf{R}(A, B, A + D_1, B + D_2) \subset P(\mathbf{H}_1 \cup \mathbf{H}_2).$$

Proof of Lemma 4.2. Let $A = A_1 + A_2$; $B = B_1 + B_2$. If $0 < u \leq D_1$ and $0 < v \leq D_2$, then $\mathbf{x}_1 = (A_1 + u, B_1) \in \mathscr{H}(A_1, D_1, B_1)$ and $\mathbf{x}_2 = (A_2, B_2 + v) \in \mathscr{V}(A_2, D_2, B_2)$. Thus

$$\mathbf{x}_1 + \mathbf{x}_2 = (A_1 + A_2 + u, B_1 + B_2 + v) = (A + u, B + v) \in P(\mathbf{H}_1) + P(\mathbf{H}_2) = P(\mathbf{H}_1 \cup \mathbf{H}_2).$$

But

$$\mathbf{R}(A, B, A + D_1, B + D_2) = \{\mathbf{x}_1 + \mathbf{x}_2 \mid 0 < u \leq D_1 \text{ and } 0 < v \leq D_2\},$$

which completes the proof. □

Proof of Theorem 4.1. Let $\mathbf{a}_1 = (0, 0)$ and let us write

$$\mathbf{a}_n = \mathbf{a}_{n-1} + \begin{cases} (0, 1) & \text{if } n = 2^k, \\ (1, 0) & \text{if } n \neq 2^k. \end{cases} \tag{4.6}$$

Let $\mathbf{A}_1 = \{\mathbf{a}_k \mid k \equiv 0 \text{ or } 3 (\operatorname{mod} 4)\}$ and $\mathbf{A}_2 = \mathbf{A} \backslash \mathbf{A}_1$.

Clearly, $\mathbf{A}_1$ satisfies (4.3); thus by Lemmas 4.1 and 4.2 we obtain (4.1). The sequence $\mathbf{A}_2$ (and hence $P(\mathbf{A}_2)$) has bounded gaps. Indeed,

$$\mathbf{a}_{4k+1} - \mathbf{a}_{4k-2} = (\mathbf{a}_{4k+1} - \mathbf{a}_{4k}) + (\mathbf{a}_{4k} - \mathbf{a}_{4k-1}) + (\mathbf{a}_{4k-1} - \mathbf{a}_{4k-2}) = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3$$

and $\mathbf{c}_i \in \{(0, 1), (1, 0)\}$ for $i = 1, 2, 3$.

We are now going to prove that $\mathbf{A}$ is not L-complete.

Assume that, contrary to the assertion, there are $\mathbf{x}_0, \mathbf{d} \in \mathbb{N}^2$ such that $L(\mathbf{x}_0, \mathbf{d}) \subset P(\mathbf{A})$. Observing that $\lim_{k\to\infty} a_k^{(2)} = \infty$, we have that, for each $k \in \mathbb{N}$,

$$|\{\mathbf{x} : \mathbf{x} \in P(\mathbf{A}), x^{(2)} < k\}| < \infty.$$

This implies that $m(\mathbf{d}) > 0$.

We claim that for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that if $\mathbf{x} \in P(\mathbf{A})$ and $x^{(1)} > n_0$, then $m(\boldsymbol{x}) < \varepsilon$. Hence, choosing ε less than $m(\mathbf{d})$, we obtain a contradiction.

By (4.6) we have that $\mathbf{a}_n = (n - [\log_2 n], [\log_2 n])$. Thus there is an $n_1 \in \mathbb{N}$ such that $m(\mathbf{a}_n) < \varepsilon/2$ for every $n > n_1$. Let $n_1 > \max\{2, 2/\varepsilon\}$ and let $n_0 = n_1^3$.

Let $\mathbf{x} \in P(\mathbf{A}), x^{(1)} > n_0$, and let us write $\mathbf{x}$ in the form

$$\mathbf{x} = \sum_{a_i \in A} \varepsilon_i \mathbf{a}_i = \sum_1 \varepsilon_i \mathbf{a}_i + \sum_2 \varepsilon_i \mathbf{a}_i$$

($\varepsilon_i = 0$ or 1) where $\sum_1$ contains those $\mathbf{a}_i$ for which $a_i^{(1)} \leq n_1$ and $\sum_2$ contains those ones for which $\mathbf{a}_i^{(1)} > n$. Let $\mathbf{x}_1 = \sum_1 \varepsilon_i \mathbf{a}_i$ and $\mathbf{x}_2 = \sum_2 \varepsilon_i \mathbf{a}_i$. Then, by (4.6), we obtain

$$x_1^{(1)} < 2 \cdot \binom{n_1}{2} < n_1^2. \tag{4.7}$$

Clearly, $m(\mathbf{x}_1) \leq 1$. Furthermore,

$$m\left(\sum_2 \varepsilon_i \mathbf{a}_i\right) < \varepsilon/2. \tag{4.8}$$

Indeed, since $m(\mathbf{a}_i) < \varepsilon/2$, i.e. $a_i^{(2)} < (\varepsilon/2)a_i^{(1)}$, we have $\sum_2 \varepsilon_i a_i^{(2)} < (\varepsilon/2) \sum_2 \varepsilon_i a_i^{(1)}$, which implies that $x_2^{(2)} < (\varepsilon/2)x_2^{(1)}$, i.e. $m(\mathbf{x}_2) < \varepsilon/2$. Now, by (4.7) and (4.8), we have

$$m(\mathbf{x}) = x^{(2)}/x^{(1)} \leq \{(x^{(1)} - n_1^2) \cdot (\varepsilon/2) + n_1^2\}/x^{(1)} < \varepsilon/2 + n_1^2/x^{(1)} < \varepsilon/2 + 2/n_1 < \varepsilon,$$

since $n_1 > 2/\varepsilon$. □

Proof of Theorem 4.2. Let $k := [\sqrt{n}/6]$. We can assume without loss of generality that $H := H(n) - H(k) > n/3$ and $a_1^{(2)} > 10$. Now we need a lemma:

LEMMA 4.3. *Let* $\mathbf{A} \subseteq \mathbb{N}^2$ *be a simple sequence. Suppose that* $\mathbf{A}$ *satisfies* (4.3). *Then there are* $B, n_0 \in \mathbb{N}$ *such that, for* $n > n_0$,

$$[B+1, B + a_1^{(2)}\sqrt{n}/40] \subset P_2(\mathbf{A}^{(k)}).$$

PROOF OF LEMMA 4.3. By (4.3) we can assume that $V(k) > \max\{10a_1^{(2)}, 5(a_1^{(2)}+1)/(a_1^{(2)}-5)\}$. Let $D := [V(k)/5]$ and $z := \min\{P \mid : V(p) = D\}$. Let us write $P_2(\mathbf{A}^{(z)}) = \{x_1 < x_2 < \cdots < x_s\}$. Clearly, $s > z \geq D$. Since $\mathbf{A}$ is a simple sequence, and by the definition of z, we have $a_z^{(2)} = a_1^{(2)} + D$. This implies that, for every i, $1 \leq i < s$,

$$x_{i+1} - x_i \leq a_1^{(2)} + D. \tag{4.9}$$

Since $V(k) - V(z) \geq 4D$, by Lemma 4.1, we can select a subset $\mathbf{A}^* = \{\mathbf{a}_{i_1}, \mathbf{a}_{i_1-1}, \ldots, \mathbf{a}_{i_{2D}}, \mathbf{a}_{i_{2D-1}} \mid \mathbf{a}_{i_j} - \mathbf{a}_{i_{j-1}} = (0,1);\quad \mathbf{a}_{i_{j-1}} \neq \mathbf{a}_{i_{j+1}}$ for $j = 1, \ldots, 2D\}$ of $\{\mathbf{a}_{z+1}, \mathbf{a}_{z+2}, \ldots, \mathbf{a}_k\}$ such that $P(\mathbf{A}^*) \supset \mathcal{V}(A, 2D, B)$ for some $A, B \in \mathbb{N}$.

In view of (4.9) and by $D > a_1^{(2)}$, we obtain that there is a $T \in \mathbb{N}$ such that

$$P_2(\mathbf{A}^{(z)}) + P_2(\mathbf{A}^*) \supset [T+1, T + (x_s - x_1)]. \tag{4.10}$$

Since $a_i^{(2)} \geq a_1^{(2)}$, we have

$$x_s - x_1 \geq a_1^{(2)} \cdot s > a_1^{(2)} \cdot z \geq a_1^{(2)} \cdot D. \tag{4.11}$$

Let

$$C := \{\mathbf{a}_{z+1}, \ldots, \mathbf{a}_k\} \setminus \mathbf{A}^* = \{\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_\mu}\}.$$

Clearly, for every ν, $1 \leq \nu \leq \mu$,

$$a_{j_\nu}^{(2)} \leq a_1^{(2)} + V(k) \leq a_1^{(2)} + 5D + 5.$$

Since $D > (a_1^{(2)}+5)/(a_1^{(2)}-5)$, and by (4.11), we obtain

$$x_s - x_1 \geq a_1^{(2)} \cdot D \geq a_1^{(2)} + 5D + 5 \geq a_{j_\nu}^{(2)}$$

for every $1 \leq \nu \leq \mu$.

This implies that

$$\begin{aligned} P_2(\mathbf{A}^{(k)}) &\supset P_2(\mathbf{A}^{(z)}) + P_2(\mathbf{A}^*) + \left\{ \sum_{r=1}^{\nu} a_{j_r} \,\middle|\, 1 \leq \nu \leq \mu \right\} \\ &\supset \left[T + 1 + a_{j_1}, T + (x_s - x_1) + \sum_{r=1}^{\mu} a_{j_r} \right], \end{aligned} \tag{4.12}$$

i.e. $P_2(\mathbf{A}^{(k)})$ contains an interval of length

$$M := (x_s - x_1) + \sum_{r=2}^{\mu} a_{j_r}^{(2)}.$$

In view of (4.11) and by $a_i^{(2)} \geq a_1^{(2)}$, we have

$$M \geq a_1^{(2)} \cdot z + a_1^{(2)} \cdot (\mu - 1) \geq a_1^{(2)} \cdot \max\{z, \mu - 1\}.$$

Since $k \leq z + 4D + \mu < 5z + \mu$, we obtain $\max\{z, \mu-1\} > (k-1)/6 \geq \sqrt{n}/40$. Thus

$$M \geq a_1^{(2)}\sqrt{n}/40. \qquad \blacksquare$$

We now turn to the proof of Theorem 4.2.

Define w by

$$H(w) - H(k) = [H/2].$$

Let us write

$$\mathbf{H}_1 := \{\mathbf{a}_i \mid k < i < w;\ \mathbf{a}_i \in \mathbf{A}\} \text{ and } \mathbf{H}_2 := \{\mathbf{a}_i \mid w \leqslant i \leqslant n;\ \mathbf{a}_i \in \mathbf{A}\}.$$

By using Lemma 4.3, we obtain that there exists an $U \in \mathbb{N}$ such that for every i, $U < i \leqslant U + M$, there is a $\mathbf{z}_{j_i} \in P(\mathbf{A}^{(k)})$ for which $z_{j_i}^{(2)} = i$.

Since $\mathbf{A}$ is a simple sequence, it is easy to see that $a_k^{(1)} \leqslant a_1^{(1)} + k$. This implies that

$$z_{j_i}^{(1)} < a_1^{(1)} \cdot k + k^2 < 2 \cdot k^2 < n/10 \tag{4.13}$$

if n is large enough. Let $\mathbf{z}_{s_1}, \mathbf{z}_{s_2}, \ldots, \mathbf{z}_{s_M}$ be a permutation of the sequence $\mathbf{z}_{j_1}, \mathbf{z}_{j_2}, \ldots, z_{j_m}$ such that

$$z_{s_1}^{(1)} \leqslant z_{s_2}^{(1)} \leqslant \cdots \leqslant z_{s_M}^{(1)}$$

holds.

By Lemma 4.1 there are $m_1, F \in \mathbb{N}$ such that

$$P(\mathbf{H}_1) \supset \mathscr{H}(F, [H/2]/2, m_1).$$

Let

$$\mathbf{y}_j = (F + 1 + z_{s_M}^{(1)} - z_{s_j}^{(1)}, m_1),$$

for $1 \leqslant j \leqslant M$. By (4.13), and since $[H/2]/2 > n/9$, we conclude that $\mathbf{y}_j \in \mathscr{H}(F, [H/2]/2, m_1)$ and

$$\mathbf{z}_{s_j} + \mathbf{y}_j = (F + z_{s_M}^{(1)}, m_1 + s_j)$$

for every $1 \leqslant j \leqslant M$.

Thus we obtain

$$\{\mathbf{z}_{s_j} + \mathbf{y}_j \mid j = 1, \ldots, M\} \subset \mathscr{V}(F + 1 + z_{s_M}^{(1)}, M, m_1).$$

Furthermore, by using Lemma 4.3 again we obtain that there are $m_2, E \in \mathbb{N}$ such that

$$\mathscr{H}(E, [H/2]/2, m_2) \subset P(\mathbf{H}_2).$$

Since the sets $\mathbf{A}^{(k)}, \mathbf{H}_1$ and $\mathbf{H}_2$ are pairwise disjoint, applying Lemma 4.2 there are $N, S \in \mathbb{N}$ such that

$$(P(\mathbf{A}^{(k)}) + P(\mathbf{H}_1)) + P(\mathbf{H}_2) \supset \mathbf{R}(N, S, N + M, S + [H/2]/2)$$

holds. This implies that

$$T(n) \geqslant M \cdot [H/2]/2 \geqslant (a_1^{(1)} \cdot \sqrt{n}/40) \cdot (n/9) = a_1^{(1)} \cdot n^{\frac{3}{2}}/360 > n^{\frac{3}{2}}/360.$$

Now we are going to construct a set $\mathbf{A}$ for which $T(n) \leqslant 4n^2$. Let $\mathbf{a}_1 = (0, 0)$ and, for every $n \in \mathbb{N}$, let

$$\mathbf{a}_{n+1} = \mathbf{a}_n + \begin{cases} (1, 0) & \text{if } n \equiv 0 \pmod 2, \\ (0, 1) & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

This definition implies that

$$|a_n^{(1)} - a_n^{(2)}| \leqslant 1.$$

Hence, if $\mathbf{x} \in P(A^{(n)})$, then

$$|x^{(1)} - x^{(2)}| \leqslant n. \tag{4.14}$$

Let us assume that, contrary to the assertion, there exists an $\mathbf{R} = \mathbf{R}(a, b, c, d) \subset P(\mathbf{A}^{(n)})$ for which $t(\mathbf{R}) > 4n^2$. This implies that

$$\max\{c - a, d - b\} > 2n$$

and thus $\max\{|b-c|, |b-a|, |d-a|\} > n$. But $\{(a, b), (c, b), (a, d)\} \subset (\mathbf{A}^{(n)})$. This contradicts (4.14) and the proof of the upper bound is completed. $\square$

PROOF OF THEOREM 4.3. First we are going to prove that there are $\mathbf{x}_1, \mathbf{x}_2, \mathbf{e}_1, \mathbf{e}_2 \in \mathbb{N}^2$ such that $m(\mathbf{e}_i) = m(\mathbf{d}_i)$ and $L(\mathbf{x}_i, \mathbf{e}_i) \subset P(\mathbf{X})$ for $i = 1, 2$.

Let $X_i \subset \mathbb{N}$, $i = 1, 2$, denote the set of integers for which

$$k \in X_i \text{ iff } k \cdot \mathbf{d}_i \in \mathbf{X} \cap L(\mathbf{d}_i). \tag{4.15}$$

This definition implies that

$$m \in P(X_i) \text{ iff } m \cdot \mathbf{d}_i \in P(\mathbf{X} \cap L(\mathbf{d}_i)). \tag{4.16}$$

Furthermore, if $k \cdot \mathbf{d}_i \in L(\mathbf{d}_i) \cap \mathbf{S}_n$, then $k \cdot |\mathbf{d}_i| \leqslant n$. Hence, by (4.4) and (4.14), we conclude that there are ε', $0 < \varepsilon' < \varepsilon$, and n_0 such that $X_i(n) > n^{\frac{1}{2}+\varepsilon'}$ for $n > n_0$. Thus, by using Theorem F, we obtain that X_i is subcomplete, i.e. there are $a_{i0}, q_i \in \mathbb{N}$ such that $\{a_{io} + k \cdot q_i\}_{k=0}^{\infty} \subset P(X_i)$. Thus, by (4.16), we have

$$\{(a_{i0} + k \cdot q_i) \cdot \mathbf{d}_i\}_{k=0}^{\infty} \subset P((\mathbf{X} \cap L(\mathbf{d}_i)).$$

Now, if $\mathbf{x}_i = a_{i0} \cdot \mathbf{d}_i$; $\mathbf{d}_i$; $\mathbf{e}_i = q_i \cdot \mathbf{d}_i$, then $L(\mathbf{x}_i, \mathbf{e}_i) \subset P(\mathbf{X} \cap L(\mathbf{d}_i))$ for $i = 1, 2$, as we claimed.

Since $m(\mathbf{d}_1) \neq m(\mathbf{d}_2)$, we have $L(\mathbf{x}_1, \mathbf{e}_1) \cap L(\mathbf{x}_2, \mathbf{e}_2) = \varnothing$, and so

$$L := L(\mathbf{x}_1, \mathbf{e}_1) + L(\mathbf{x}_2, \mathbf{e}_2) \subset P(\mathbf{X}).$$

Let $\mathbf{x}_0 = \mathbf{x}_1 + \mathbf{x}_2$. Clearly, $L = \{\mathbf{x}_0 + k \cdot \mathbf{e}_1 + m \cdot \mathbf{e}_2 \mid k, m \in \mathbb{N}\}$. Let $r > |\mathbf{e}_1| + |\mathbf{e}_2|$ and $r = s$. Since the set $\mathbf{A} = \mathbf{X} \setminus (L(\mathbf{d}_1) \cup L(\mathbf{d}_2))$ satisfies (4.3), by Lemmas 4.1 and 4.2 there exist integers a and b such that $\mathbf{R}(a, b, a+r, b+s) \subset P(\mathbf{A})$. Let $\mathbf{w} = (a, b)$ and $\mathbf{y}_0 = \mathbf{x}_0 + \mathbf{w}$. We are going to show that if $\mathbf{z} \in \mathbf{S}(\mathbf{y}_0, \mathbf{e}_1, \mathbf{e}_2)$, then $\mathbf{z} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in L$ and $\mathbf{v} \in P(\mathbf{A})$. Define the integers k and m by

$$m(\mathbf{e}_1) \leqslant m(\mathbf{z} - (\mathbf{y}_0 + k \cdot \mathbf{e}_1 + m \cdot \mathbf{e}_2) \leqslant m(\mathbf{e}_2)$$

and

$$|\mathbf{z} - (\mathbf{y}_0 + k \cdot \mathbf{e}_1 + m \cdot \mathbf{e}_2)| = \min\{|\mathbf{z} - (\mathbf{y}_0 + s \cdot \mathbf{e}_1 + t \cdot \mathbf{e}_2| : s, t \in \mathbb{N}\}.$$

Since $|\mathbf{z} - (\mathbf{y}_0 + k \cdot \mathbf{e}_1 + m \cdot \mathbf{e}_2)| < |\mathbf{e}_1| + |\mathbf{e}_2| < r$, we obtain that $\mathbf{v} = \mathbf{z} - (\mathbf{y}_0 + k \cdot \mathbf{e}_1 + m \cdot \mathbf{e}_2) \in \mathbf{R}(a, b, a+r, b+r)$. Let $\mathbf{u} = \mathbf{y}_0 + k \cdot \mathbf{e}_1 + m \cdot \mathbf{e}_2$. Thus $\mathbf{u} + \mathbf{v} = \mathbf{z}$, $\mathbf{u} \in L$, $\mathbf{v} \in P(\mathbf{A})$. So we obtain that $\mathbf{S}(\mathbf{y}_0, \mathbf{e}_1, \mathbf{e}_2) \subset P(\mathbf{X})$, i.e. $\mathbf{X}$ is p-complete. $\square$

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